Section 15.9:
Change of Variables
In Multiple Integrals

What We'll Learn In Section 15.9

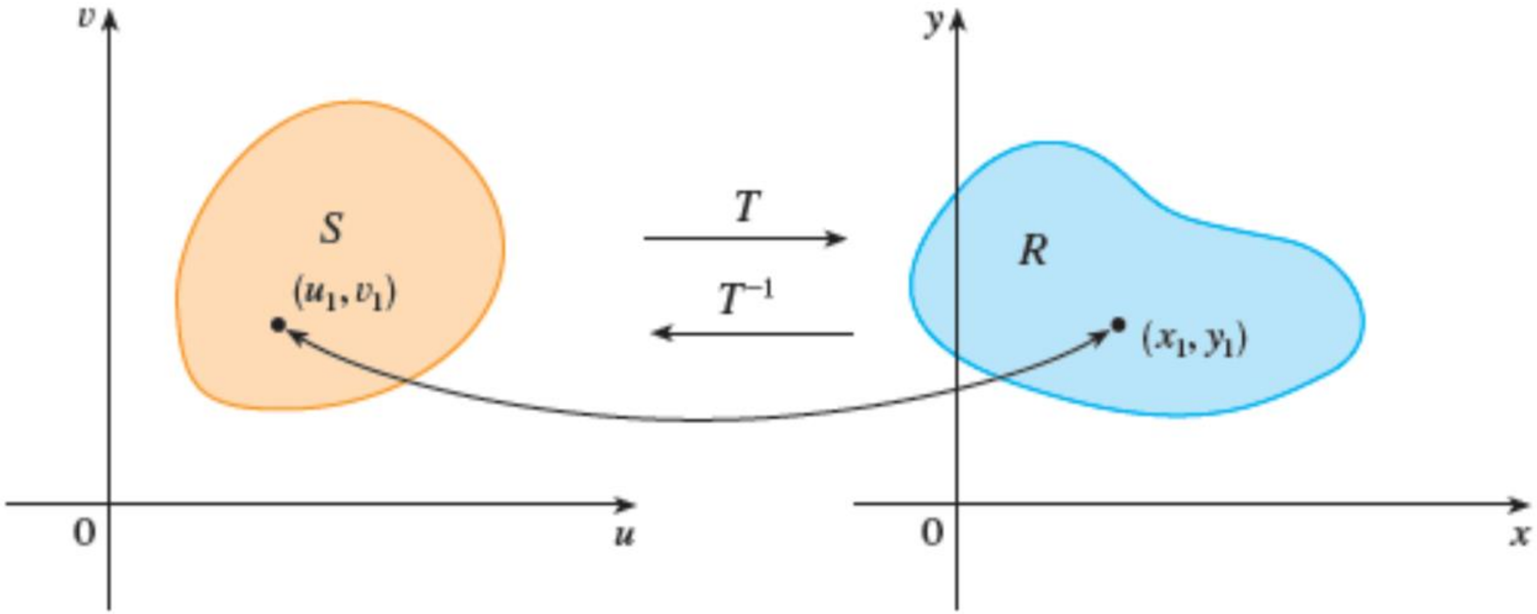
1. Transformations
2. The Jacobian
3. Change of Variables in a Double & Triple Integral

1. Transformations

Idea of Sec. 15.9:

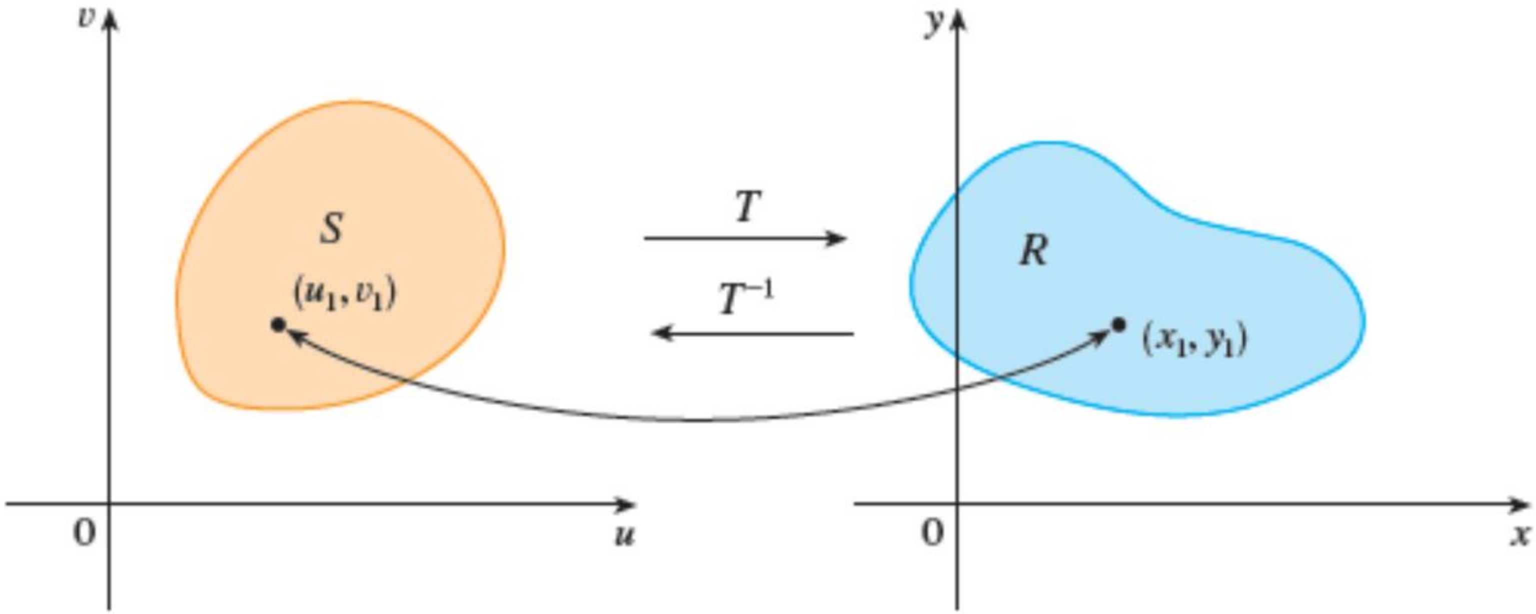
- Imagine we are trying to calculate a double (or triple) integral and we don't like the region we are integrating over.
- We can change the region we are integrating over (R) to a better region (S) by applying a transformation (or a change of variables)

1. Transformations



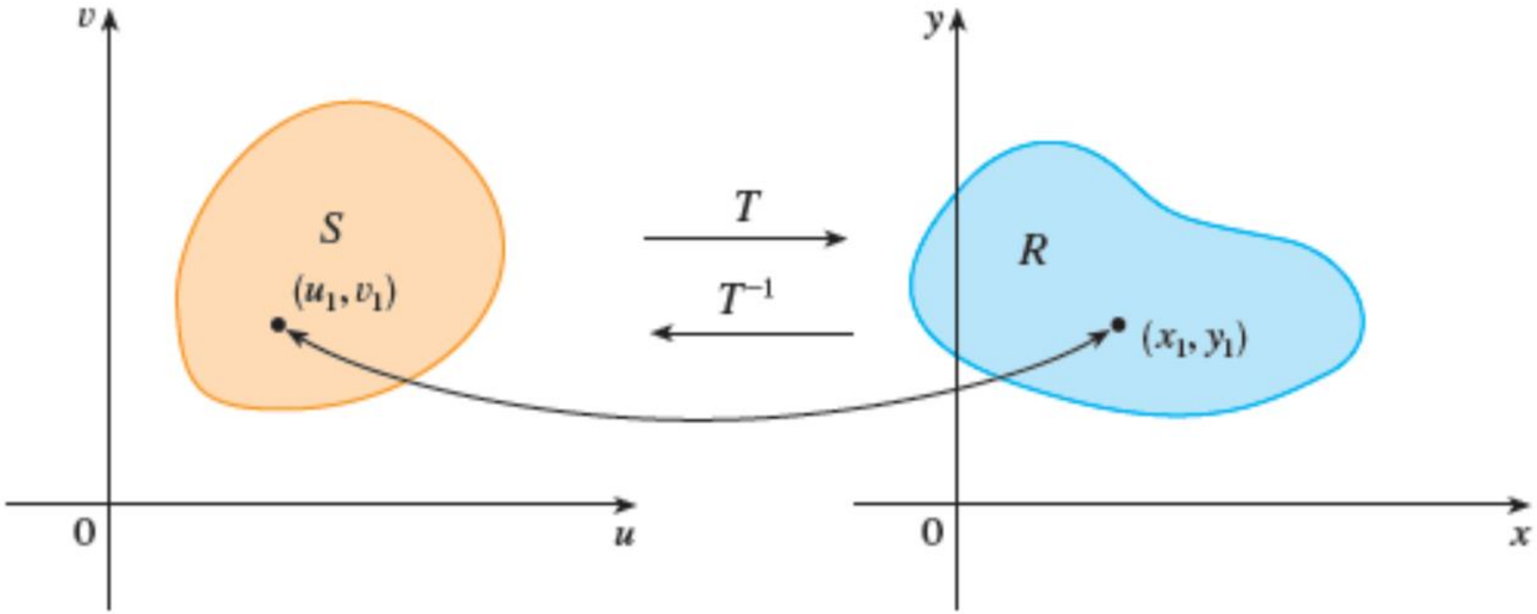
- A transformation T is a function from $\mathbb{R}^2$ to $\mathbb{R}^2$.
- We think of it as a function from the *uv*-plane to the *xy*-plane.

1. Transformations



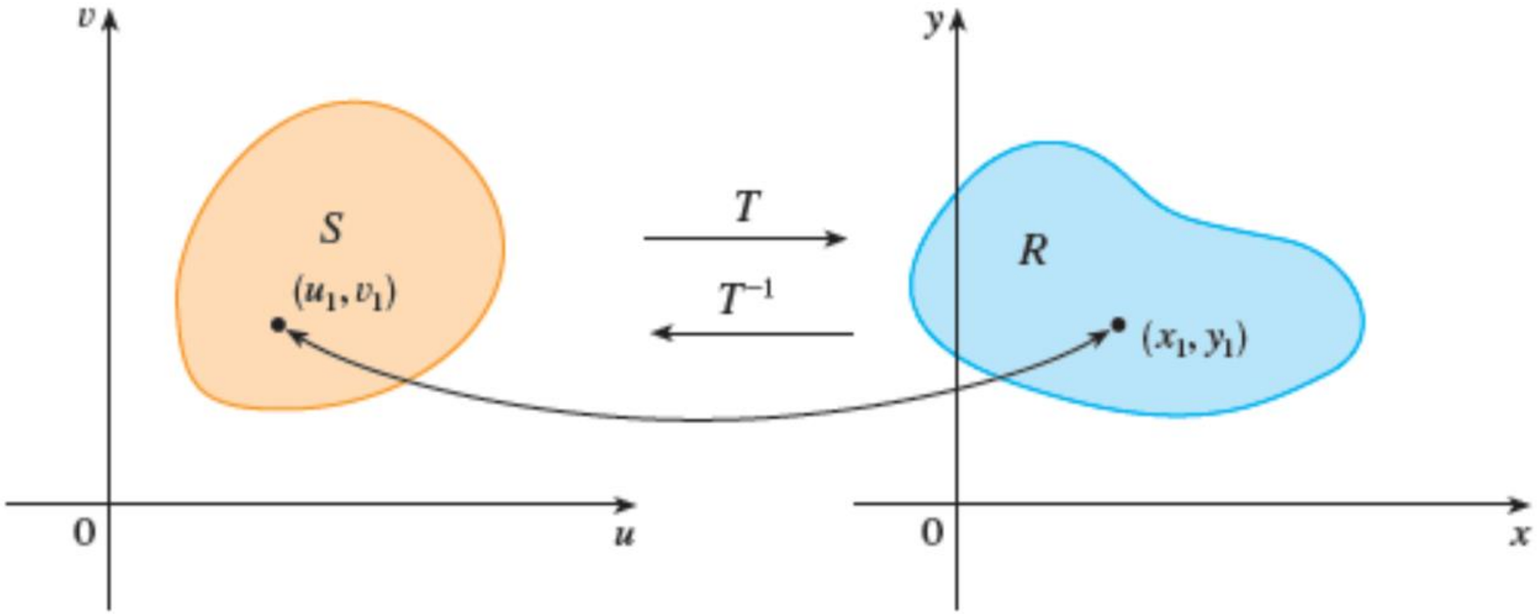
- Given an input (u_1, v_1) of T, its <u>image</u> is its output $T(u_1, v_1)$, which is the point (x_1, y_1) above.
- The <u>image of S</u> is the collection of all outputs when all points from S are plugged into T, which is R above.

1. Transformations



- T is <u>one-to-one</u> if different inputs have different outputs (images).
- If T is one-to-one, then it has an <u>inverse transformation</u> T^{-1} where S would be the image of R.

1. Transformations



- Usually T is given as 2 equations

$$x = g(u, v) \text{ and } y = h(u, v)$$

- If T has an inverse, then these equations can be solved for u and v

$$u = G(x, y) \text{ and } v = H(x, y)$$

1. Transformations

Ex 1: A transformation is defined by the equations

$$x = u^2 - v^2 \quad , \quad y = 2uv.$$

Find the image of the square

$$S = \{(u, v) \mid 0 \leq u \leq 1 , 0 \leq v \leq 1 \}$$

2. The Jacobian

7 Definition

The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

For 3 variable transformations (or triple integrals)…

$$\frac{\partial (x, y, z)}{\partial (u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

3. Change of Variables in a Double & Triple Integrals

9 Change of Variables in a Double Integral

Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x,y)\,dA = \iint_S f\left(x\left(u,v\right), y\left(u,v\right)\right) \left| \frac{\partial\left(x,y\right)}{\partial\left(u,v\right)} \right| du\,dv$$

3. Change of Variables in a Double & Triple Integrals

Change of Variables in a Triple Integral…

$$\iiint_R f(x, y, z)\, dV$$

$$= \iiint_S f\left(x\left(u, v, w\right), y\left(u, v, w\right), z\left(u, v, w\right)\right) \left| \frac{\partial\left(x, y, z\right)}{\partial\left(u, v, w\right)} \right| du\, dv\, dw$$

3. Change of Variables in a Double & Triple Integrals

<u>Ex 2</u>: Use the change of variables

$$x = u^2 - v^2 \quad , \quad y = 2uv$$

to evaluate the integral $\iint_R y\, dA$, where R is the region bounded by the *x*-axis and the parabola $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.

3. Change of Variables in a Double & Triple Integrals

Ex 3:

Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\, dA$, where R

Is the trapezoidal region with vertices (1,0) , (2,0) , (0,-2) , and (0,-1).

3. Change of Variables in a Double & Triple Integrals

Ex 4: Derive the formula for triple integration in spherical coordinates.